

# Interacting Particles and Strings in Path and Surface Representations.

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## Abstract

Non-relativistic charged particles and strings coupled with abelian gauge fields are quantized in a geometric representation that generalizes the Loop Representation. We consider three models: the string in self-interaction through a Kalb-Ramond field in four dimensions, the topological interaction of two particles due to a BF term in  $2+1$  dimensions, and the string-particle interaction mediated by a BF term in  $3+1$  dimensions. In the first case one finds that a consistent "surface-representation" can be built provided that the coupling constant is quantized. The geometrical setting that arises corresponds to a generalized version of the Faraday's lines picture: quantum states are labeled by the shape of the string, from which emanate "Faraday's surfaces". In the other models, the topological interaction can also be described by geometrical means. It is shown that the open-path (or open-surface) dependence carried by the wave functional in these models can be eliminated through an unitary transformation, except by a remaining dependence on the boundary of the path (or surface). These feature is closely related to the presence of anomalous statistics in the  $2+1$  model, and to a generalized "anyonic behavior" of the string in the other case.

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## I. INTRODUCTION

In this paper we study Abelian theories of interacting non-relativistic point particles and strings, and quantize them in geometrical representations that generalize the Loop Representation (LR)[1]. The interactions are mediated by Abelian gauge fields, and special attention is devoted to topological interactions.

The first model we deal with is that of a string self-interacting through a Kalb-Ramond field [2]. This study is a generalization of that of reference [3], where charged non-relativistic point particles in electromagnetic interaction were quantized within the LR [1] framework. In [3] it was found that charge must be quantized in order to the LR formulation of the model be consistent. This result agrees with those obtained in previous developments [4, 5].

In this work we find that the coupling constant of the string (lets say, the Kalb-Ramond “charge” of the string) must be quantized, also, if the geometric representation adapted to the model is consistent.

Following [3], we also consider the LR in the case of topological interactions. Then, as a second model, we study the theory of two kinds of non-relativistic point particles, in  $2 + 1$ -dimensions, interacting through a BF term. One may couple minimally the first type of particles to one of the vector fields and the second type to the other one. Using this model one prevents self-interaction problems that arises in the model of particles interacting by means of the Chern-Simons field [3]. This fact leads us to see that in the case of just two particles, the model precisely corresponds to the quantum mechanical system of two anyons, which has the virtue of being exactly soluble [6, 7, 8, 9]. Furthermore, this “toy-model” opens the way and gives us the key for understanding the more involved theory of the next section.

This last model consists of a particle and a string interacting through a BF topological term in  $3 + 1$ -dimensions. Again, the particle couples in a natural way with the 1-form ( $A_\mu$ ), while the string does it with the 2-form ( $B_{\mu\nu}$ ). The quantization can be done in two “dual” geometric frameworks: a path and a surface representation.

As in references [3, 11], when the fields that provide the interaction have a topological character, the dependence of the wave-functionals on paths (in general, on the appropriate geometric objects that enter in the representation, like paths or surfaces) may be eliminated by means of an unitary transformation. In that case one obtains a quantum mechanics

of particles, or particles and strings (depending on the model), subjected to a long range interaction.

As in the particle-field interaction [3], the coupling of extended “matter” objects to fields presents certain subtleties regarding its quantization, and so does the appropriate geometric representation. We shall deal with an extension of the conventional LR, namely, the surface representation, which was considered several years ago to study the free-field case [12, 13], but has to be adapted to include the particularities that the coupling with the string demands

The paper is organized as follows. In section II we study a geometric surface representation for the non-relativistic “charged” string in Kalb-Ramond interaction. Section III is dedicated to consider the path representation quantization of two different species of non-relativistic point particles interacting by means of a topological BF term in 2+1-dimensions. We devote section IV to the study of the interaction of non-relativistic point charged particles and strings through a topological BF term in 3 + 1-dimensions. Some discussions and final remarks are given in the last section.

## II. NON-RELATIVISTIC STRING INTERACTING WITH THE KALB-RAMOND FIELD. SURFACE-REPRESENTATION.

The first model we are going to study is described by the action

$$S = -\frac{1}{12g^2} \int H^{\mu\nu\lambda} H_{\mu\nu\lambda} d^4x + \frac{\alpha}{2} \int dt \int d\sigma \left[ (\dot{z}^i)^2 - (z'^i)^2 \right] + \frac{1}{2} \int d^4x J^{\mu\nu} B_{\mu\nu} \quad (1)$$

Although the theory is not Lorentz invariant, we find it convenient to employ four-space notation to some extent. The Kalb-Ramond antisymmetric potential and field,  $B_{\mu\nu}$  and  $H_{\mu\nu\lambda}$ , are related by  $H_{\mu\nu\lambda} = 3\partial_{[\mu} B_{\nu\lambda]} = \partial_\mu B_{\nu\lambda} + \partial_\lambda B_{\mu\nu} + \partial_\nu B_{\lambda\mu}$ . Besides the Kalb-Ramond term (the first one), the action comprises a contribution corresponding to the free non-relativistic closed string, whose world sheet spatial coordinates  $z^i(t, \sigma)$  are given in terms of the time  $t$  and the parameter  $\sigma$  along the string (dot and prime indicate derivatives with respect to  $t$  and  $\sigma$  respectively).  $\alpha$  is the string tension, having units of  $mass^2$  and  $g$  is a parameter with units of  $mass$  in order to have a dimensionless coupling constant between the string and the Kalb-Ramond potential. In the string-field interaction term, the current

$J^{\mu\nu}$  associated to the string is given by

$$J^{\mu\nu}(\vec{x}, t) = \phi \int dt \int d\sigma [\dot{z}^\mu z'^\nu - \dot{z}^\nu z'^\mu] \delta^{(4)}(x - z), \quad (2)$$

where  $\phi$  is the dimensionless coupling constant. This current will be dynamically conserved. The interaction term can be written as

$$S_{int} = \frac{\phi}{2} \int dt \int d\sigma [\dot{z}^\mu z'^\nu - \dot{z}^\nu z'^\mu] B_{\mu\nu}(z). \quad (3)$$

The generalization of what we are going to study to the case of more than one string, even with different couplings for each one, is straightforward. For the sake of simplicity we shall mainly consider the model with just one string; some remarks about the general case are given at the end of this section. The action is invariant under the gauge transformations

$$\delta B_{\mu\nu} = 2\partial_{[\mu}\lambda_{\nu]} = \partial_\mu\lambda_\nu - \partial_\nu\lambda_\mu, \quad (4)$$

provided the string is closed. We are interested in performing the Dirac quantization of the theory. To this end, we need the 3 + 1 decomposition of the action [we are employing the “metric”  $\eta_{\mu\nu} = (+, -, -, -)$ ]

$$S = \int d^4x \left( -\frac{1}{12g^2} H_{ijk} H_{ijk} + \frac{1}{4g^2} H_{0ij} H_{0ij} + B_{0i} J^{0i} + \frac{1}{2} B_{ij} J^{ij} \right) + \frac{\alpha}{2} \int dt \int d\sigma [(\dot{z}^i)^2 - (z'^i)^2], \quad (5)$$

so the conjugate momenta associated to the fields,  $B_{ij}$ , and string variables,  $z^i$ , are

$$\Pi^{ij} = \frac{1}{2g^2} (\dot{B}_{ij} + \partial_j B_{0i} - \partial_i B_{0j}), \quad P_i = \alpha \dot{z}^i + \phi B_{ij} z'^j. \quad (6)$$

The field variables  $B_{i0}$ , which have vanishing momenta, are treated as non-dynamical fields from the very beginning. In fact, the Hamiltonian results to be

$$H = \int d^3x \left[ g^2 \Pi^{ij} \Pi^{ij} + \frac{1}{12g^2} H_{ijk} H_{ijk} \right] + \int d\sigma \frac{\alpha}{2} \left[ \frac{1}{\alpha^2} (P_i - \phi B_{ij}(z) z'^j)^2 + (z'^i)^2 \right] + \int d^3x B_{0i} \chi^i, \quad (7)$$

hence, the role of  $B_{i0}$  as Lagrange multipliers enforcing the constraints

$$\chi^i(x) \equiv -\rho^i(x) - 2\partial_j \Pi^{ji}(x) = 0, \quad (8)$$

with

$$\rho^i(x) \equiv \phi \int d\sigma z'^i \delta^{(3)}(\vec{x} - \vec{z}), \quad (9)$$

becomes evident.

The canonical Poisson brackets are defined as

$$\{z^i(\sigma), P_j(\sigma')\} = \delta_j^i \delta(\sigma - \sigma'), \quad (10)$$

$$\{B_{ij}(\vec{x}), \Pi^{kl}(\vec{y})\} = \frac{1}{2} (\delta_i^k \delta_j^l - \delta_i^l \delta_j^k) \delta^{(3)}(\vec{x} - \vec{y}). \quad (11)$$

The remaining Poisson brackets vanish.

The preservation of the constraints given above does not produce new ones. Furthermore, they result to be first class constraints that generate time independent gauge transformations on the phase space of the theory.

The basic observables in the sense of Dirac that can be constructed from the canonical variables are the generalized electric and magnetic fields

$$\Pi^{ij} = \frac{1}{2g^2} H_{0ij} \equiv \frac{1}{2g^2} E^{ij}, \quad (12)$$

$$\mathbf{B} \equiv \frac{1}{3!} \epsilon^{ijk} H_{ijk}, \quad (13)$$

the position  $z^i(\sigma)$ , and the covariant momentum of the string

$$P_i - \phi B_{ij}(z) z'^j. \quad (14)$$

All the physical observables of the theory are built in terms of these gauge invariant quantities, as can be verified. For instance, the Hamiltonian, given in equation (7) fulfils this requirement.

To quantize, the canonical variables are promoted to operators obeying the commutators that result from the replacement  $\{, \} \longrightarrow -i[, ]$ . These operators have to be realized in a Hilbert space of physical states  $|\Psi\rangle_{Phys}$ , that obey the generalized Gauss law

$$-(\rho^i(x) + 2\partial_j \Pi^{ji}(x)) | \Psi_{Physical} \rangle \approx 0. \quad (15)$$

At this point, we introduce a geometric representation adapted to the present model. It will be an “open-surfaces representation”, which is closely related with the LR as formulated by Gambini and Trías [1], and with an early geometrical formulation of the pure Kalb-Ramond field, based on closed surfaces [12, 13].

Consider the space of piecewise smooth oriented surfaces (for our purposes) in  $R^3$ . A typical element of this space, let say  $\Sigma$ , will be the union of several surfaces, perhaps some

of them being closed. In this space we set up the following equivalence relation: we identify two  $\Sigma$ 's that share the same "form factor"  $T^{ij}(x, \Sigma)$  defined as

$$T^{ij}(x, \Sigma) = \int d\Sigma_y^{ij} \delta^{(3)}(\vec{x} - \vec{y}). \quad (16)$$

with  $d\Sigma_y^{ij} = (\frac{\partial y^i}{\partial s} \frac{\partial y^j}{\partial r} - \frac{\partial y^i}{\partial r} \frac{\partial y^j}{\partial s}) ds dr$ ,  $s, r$  being parameters for the surface. It is easy to show that this indeed defines an equivalence relation. Also, observe that two surfaces differing in the parametrization belong to the same class, since they trivially have the same form factor.

It is worth noticing that the composition of surfaces, together with the equivalence relation stated above, define a group product among the classes. The resulting group is Abelian, since the form factor of the composition of two  $\Sigma$ 's is the sum of their respective form factors. All these features of the "open surfaces space", are more or less immediate generalizations of aspects already encountered in its one dimensional relative, the Abelian path space [1, 3, 11, 14, 15].

Now we consider functionals  $\Psi(\Sigma)$  depending on classes  $\Sigma$  [we employ the same notation both for the surface and the class to which it belongs, since from now on all the surface-dependent objects that will appear are indeed class-dependent ones]. We introduce the surface derivative  $\delta_{ij}(x)$ , that measures the response of  $\Psi(\Sigma)$  when an element of surface whose infinitesimal area is  $\sigma_{ij}$  is attached to the argument  $\Sigma$  of  $\Psi(\Sigma)$  at the point  $x$ , up to first order in  $\sigma_{ij}$

$$\Psi(\delta\Sigma \cdot \Sigma) - \Psi(\Sigma) = \sigma^{ij} \delta_{ij}(x) \Psi(\Sigma) \quad (17)$$

where

$$\sigma^{ij} = u^i v^j - v^j u^i, \quad (18)$$

is the surface element generated by the infinitesimal vectors  $\vec{u}$  and  $\vec{v}$ . The surface derivative  $\delta_{ij}(x)$  should not be confused with the loop derivative  $\Delta_{ij}(x)$ . Unlike the former, the latter acts onto loop-dependent functionals. Of course, since in  $R^3$  loop-dependence is a particular case of surface-dependence (a loop can be seen as the boundary of an open surface, whenever the manifold be trivial in the homological sense), the loop derivative can be taken as the surface derivative restricted to loop-dependent functionals. In this sense it can be said that  $\delta_{ij}(x)$  "includes"  $\Delta_{ij}(x)$ .

From  $\delta_{ij}(x)$  it is possible to define the closed-surface derivative  $\Delta_{ijk}$  of reference [13], that appends a small cube of volume  $V^{ijk}$  to the argument  $\Sigma$  of  $\Psi(\Sigma)$ . The relation between

both derivatives is

$$\Delta_{ijk}(x) = \partial_i \delta_{jk}(x) + \partial_j \delta_{ki}(x) + \partial_k \delta_{ij}(x). \quad (19)$$

It should be noticed that the right hand side of the above equation would vanish if  $\delta_{ij}(x)$  be the same as  $\Delta_{ij}(x)$ . This is so, since  $\Delta_{ij}(x)$  is the curl of a more basic object: the path-derivative (see equation 42) [13].

Turning back to the quantization of our model, it can be seen that the fundamental commutator associated to equation (11) can be realized on surface-dependent functionals if one prescribes

$$\hat{\Pi}^{ij}(\vec{x}) \longrightarrow \frac{1}{2} T^{ij}(\vec{x}, \Sigma), \quad (20)$$

$$\hat{B}_{ij}(\vec{x}) \longrightarrow 2i \delta_{ij}(\vec{x}), \quad (21)$$

since the surface-derivative of the form factor is given by

$$\delta_{ij}(\vec{x}) T^{kl}(\vec{y}, \Sigma) = \frac{1}{2} (\delta_i^k \delta_j^l - \delta_i^l \delta_j^k) \delta^{(3)}(\vec{x} - \vec{y}). \quad (22)$$

On the other hand, the operators associated to the string can be realized in a “shape” representation, i.e., onto functionals  $\Psi[z(\sigma)]$  that depend on the (spatial) coordinates of the string world sheet

$$\hat{z}^i(\sigma) \longrightarrow z^i(\sigma), \quad \hat{P}^i(\sigma) \longrightarrow -i \frac{\delta}{\delta z^i(\sigma)}. \quad (23)$$

Henceforth, the states of the interacting theory can be taken as functionals  $\Psi[\Sigma, z(\sigma)]$  depending both on surfaces (i.e. the equivalence classes discussed above) and functions  $z(\sigma)$ . Among these functionals, we must pick out those that belong to the kernel of the Gauss constraint (15), that in this representation can be written as

$$\begin{aligned} & \left( \rho^i(\vec{x}) + 2\partial_j \Pi^{ji}(\vec{x}) \right) \Psi[\Sigma, z(\sigma)] \approx 0 \implies \\ & \left( \phi \int_{string} d\sigma z'^i \delta^{(3)}(\vec{x} - \vec{z}) - \int_{\partial\Sigma} d\sigma z'^i \delta^{(3)}(\vec{x} - \vec{z}) \right) \Psi[\Sigma, z(\sigma)] \approx 0, \end{aligned} \quad (24)$$

where we have used

$$\partial_j T^{ji}(\vec{x}, \Sigma) = -T^i(\vec{x}, \partial\Sigma) = - \int_{\partial\Sigma} dz^i \delta^{(3)}(\vec{x} - \vec{z}), \quad (25)$$

with  $\partial\Sigma$  being the boundary of  $\Sigma$ . To solve this constraint, it is useful to recall which is the geometrical setting that allows to solve the Gauss constraint in Maxwell theory coupled

to non-relativistic particles, which was discussed in reference [3]. There, the appropriate physical space can be labelled by lines of Faraday: every particle carries a bundle of lines emanating from or arriving to the particle (depending on the sign of the particle's charge). This construction is possible only if charge is quantized, since the number of Faraday lines, which must be equal to the charge to which they are attached, has to be an integer. In the present case, we see that if the surface is such that its boundary coincides with the string, the constraint (24) reduces to

$$(\phi - 1) \int_{string} d\sigma z'^j \delta^{(3)}(\vec{x} - \vec{z}) = 0, \quad (26)$$

and it is satisfied for  $\phi = 1$ . In that case it can be said that the surface emanates from the string. It could well happen that, instead, the boundary of the surface had the opposite orientation of the string. Then, the constraint would demand that  $\phi = -1$ , and we say that the surface ends at the string. Clearly, there is also the possibility that the surface be composed by several layers, say  $n$  of them, that start (or end) at the string. In this situation, equation (24) becomes

$$(\phi - n) \int_{string} d\sigma z'^j \delta^{(3)}(\vec{x} - \vec{z}) = 0, \quad (27)$$

and the coupling constant must obey  $\phi = n$ . The sign of  $n$  depends on whether the layers are "incoming" or "outgoing", in the sense explained above. Finally, it should be remarked that when  $\phi = n$ , the surface may consist of the  $n$  layers attached to the string, plus an arbitrary number of closed surfaces, since the latter do not contribute to the boundary of the surface that define the equivalence class  $\Sigma$ .

Thus, we find that the physical sector of the Hilbert space of the theory, in the surface-representation, consists of wave functionals that depend on "surfaces of Faraday" for the string-Kalb-Ramond system. Notice that in the case of  $N$  strings, carrying different "charges"  $\phi_a$ ,  $a = 1, \dots, N$ , each string must be a source or sink of its own bundle of  $n_a = \phi_a$  layers (as before, these bundles may be accompanied by closed pieces of surfaces). This geometrical setting is possible if the couplings  $\phi_a$  are quantized, since each individual sheet or layer carries a unit of Kalb-Ramond electric flux.

To conclude this section, let us write down the Schrödinger equation of the model in the surface-representation

$$-i \frac{\partial}{\partial t} \Psi[\Sigma, z(\sigma)] = H \Psi[\Sigma, z(\sigma)]$$



$$\begin{aligned}
&= \frac{1}{2g^2} \int d^3x \left[ \mathbf{B}^2 + \frac{1}{2} E^{ij} E^{ij} \right] + \\
&+ \int d\sigma \frac{\alpha}{2} \left[ \frac{-1}{\alpha^2} \left( \frac{\delta}{\delta z^i} + 2\phi \delta_{ij}(\vec{z}) z'^j \right)^2 + (z'^i)^2 \right] \Psi[\Sigma, z(\sigma)]. \quad (28)
\end{aligned}$$

The first term correspond to the free-field contributions to the energy of the system. In fact,  $\mathbf{B}^2$  is a kind of "surface laplacian", while  $E^{ij} E^{ij}$  (which indeed contains a square of Dirac-delta-functions, hence it should be regularized) may be though as the "position of the surface" squared. The remaining term correspond to the string energy, taking into account the minimal coupling to the Kalb-Ramond field. It should be noticed that every term in the right hand side of this expression respects the geometrical properties of the physical sector that we have studied in the previous discussion. For instance, the covariant momentum  $-i \left( \frac{\delta}{\delta z^i} + 2\phi \delta_{ij}(\vec{z}) z'^j \right)$ , which encodes the field-string interaction, acts onto the wave functionals  $\Psi[\Sigma, z(\sigma)]$  as a generalized Mandelstam derivative [16]: while the functional derivative with respect to  $z^i(\sigma)$  translates (infinitesimally) the string, the surface derivative evaluated at the string coordinate  $\sigma$ , times  $\phi$ , serves to join the infinitesimally translated string to the bundle of layers that, otherwise, would remain separated of the string, breaking gauge invariance.

### III. TOY MODEL: NON-RELATIVISTIC PARTICLES INTERACTING THROUGH A BF TERM IN 2 + 1 DIMENSIONS

In this section we shall study the path representation of a BF term in 2 + 1 dimensions coupled with two types of dynamical particles. This "toy model" already exhibits many of the features that we shall encounter in section IV, where we shall deal with a 3+1-dimensions topologically interacting particle-string model.

The action that we shall take is written as

$$S = \frac{1}{2} \int \epsilon^{\mu\nu\lambda} B_\mu F_{\nu\lambda} d^3x + \int dt \left( \frac{1}{2} m \dot{\vec{r}}^2 + \frac{1}{2} M \dot{\vec{R}}^2 \right) + \int d^3x (j^\mu A_\mu + J^\mu B_\mu), \quad (29)$$

where  $F_{\nu\lambda} = \partial_\nu A_\lambda - \partial_\lambda A_\nu$ . We use small and capital letters to distinguish the quantities related with the two types of particles. The idea we have in mind is to generalize this model in the next section replacing the "big" particles by an extended object (string). The current

$j^\mu$ , is given by

$$j^\mu(\vec{x}) = qv^\mu\delta^{(2)}(\vec{x} - \vec{r}) = (\rho(\vec{x}), \vec{j}(\vec{x})), \quad (30)$$

where  $q$  is the charge of the "type one" particle coupled with the 1-form  $A_\mu$  and  $v^\mu = (1, \dot{\vec{r}})$  its velocity. A similar expression holds for  $J^\mu$ , with capital letters replacing small ones. The dimensions of  $A_\mu$  and  $B_\mu$  are  $length^{-1}$  so  $q$  and  $Q$  are dimensionless.

It should be observed that the BF term can be decoupled in two Chern-Simons terms via the field transformation  $A_\mu \approx a_\mu^1 + a_\mu^2$ ,  $B_\mu \approx a_\mu^1 - a_\mu^2$ . Nevertheless, when the source and particle terms are present the system does not decouple directly in two particle-Chern-Simons models.

The 2 + 1 decomposition of the action is given by

$$\begin{aligned} S = & \int d^3x \left( \epsilon^{ij} \partial_i A_j B_0 + \epsilon^{ij} \partial_i B_j A_0 + \dot{A}_j \epsilon^{ij} B_j \right) + \int dt \left( \frac{1}{2} m \dot{\vec{r}}^2 + \frac{1}{2} M \dot{\vec{R}}^2 \right) \\ & + \int dt \left( q A_0(\vec{r}) + q A_i \dot{r}^i \right) + \int dt \left( Q B_0(\vec{R}) + Q B_i \dot{R}^i \right). \end{aligned} \quad (31)$$

To perform the Dirac quantization of the model we will not take  $A_0, B_0$  as true dynamical variables. Moreover, the decomposition (32) shows that  $\epsilon^{ij} B_j$  is the conjugated momentum of  $A_i$ , and there is no need to treat  $B_i$  as an independent "generalized coordinate" [17], instead we will take  $\frac{\partial \mathcal{L}}{\partial A_i} \equiv \Pi^i = \epsilon^{ij} B_j$  as a definition. Hence, the conjugate momenta associated to the fields and particles variables are

$$\Pi^i = \epsilon^{ij} B_j, \quad p_i = m \dot{r}^i + q A_i(\vec{r}), \quad (32)$$

$$P_i = M \dot{R}^i + Q B_i(\vec{r}). \quad (33)$$

The Hamiltonian has the form

$$\begin{aligned} H = & \frac{(\vec{P} - q\vec{A}(\vec{r}))^2}{2m} + \frac{(\vec{P} - Q\vec{B}(\vec{R}))^2}{2M} + \int d^2x [A_0(x)\chi_1(x) + B_0(x)\chi_2(x)] \\ \equiv & H_0 + \int d^2x [A_0(x)\chi_1(x) + B_0(x)\chi_2(x)], \end{aligned} \quad (34)$$

where we have defined  $\chi_1(x)$  and  $\chi_2(x)$  as

$$\begin{aligned} \chi_1(x) & \equiv -\epsilon^{ij} \partial_i B_j + q \delta^{(2)}(\vec{x} - \vec{r}) \equiv \mathbf{B}_{\vec{B}} - \rho_1(x), \\ \chi_2(x) & \equiv -\epsilon^{ij} \partial_i A_j - Q \delta^{(2)}(\vec{x} - \vec{R}) \equiv \mathbf{B}_{\vec{A}} - \rho_2(x). \end{aligned} \quad (35)$$

In (34), the role of  $A_0(x)$  and  $B_0(x)$  as Lagrange multipliers that enforce the secondary constraints  $\chi_1(x) = 0$  and  $\chi_2(x) = 0$  becomes clear. In the last equations, we have defined the magnetic fields associated to  $A_\mu$  and  $B_\mu$

$$\mathbf{B}_{\vec{A}} = -\frac{1}{2}\epsilon^{ij}F_{ij} = -\epsilon^{ij}\partial_i A_j, \quad \mathbf{B}_{\vec{B}} = -\epsilon^{ij}\partial_i B_j. \quad (36)$$

It should be recalled that the BF term, being topological, does not contribute to the energy-momentum tensor. That is why the Hamiltonian  $H_0$  has the form of that of a collection of two sets of particles in different external fields. This feature also appears when dealing with the theory of particles interacting through a Chern-Simons field [3].

The canonical commutators are defined as

$$\begin{aligned} [r^i, p_j] &= i\delta_j^i, \\ [R^i, P_j] &= i\delta_j^i, \\ [A_i(\vec{x}), \epsilon^{jk}B_k(\vec{y})] &= i\delta_i^j\delta^{(2)}(\vec{x} - \vec{y}). \end{aligned} \quad (37)$$

The remaining ones vanish identically. The constraints  $\chi_1(x)$  and  $\chi_2(x)$ , written in (35), are readily seen to be of first class.

The Gauge invariant observables (in Dirac's sense) that can be constructed from the canonical variables are the generalized magnetic fields defined in (4), the positions  $\vec{r}$ ,  $\vec{R}$  and the "covariant" momenta  $p_i - qA_i(\vec{r})$  and  $P_i - QB_i(\vec{R})$ . As before, all the physical observables of the theory are built in terms of these gauge invariant quantities, as can be easily verified. These fundamental observables have to be realized in a Hilbert space of physical states  $|\psi\rangle_{Physical}$ , that obey two generalized Gauss laws (one for each type of particle) given by

$$\begin{aligned} \chi_1(x) |\psi_{Physical}\rangle &= -(\rho_1(x) + \epsilon^{ij}\partial_i B_j(x)) |\psi_{Physical}\rangle \approx 0, \\ \chi_2(x) |\psi_{Physical}\rangle &= -(\rho_2(x) + \epsilon^{ij}\partial_i A_j(x)) |\psi_{Physical}\rangle \approx 0. \end{aligned} \quad (38)$$

At this point, we introduce a geometric representation adapted to the present model. It is the Abelian path representation [1, 3, 11, 14, 15], that can be summarized as follows. Consider the space of oriented open paths in  $R^2$ . An element  $\gamma$  of this space will be the union of several curves, perhaps some of them being closed. As we did for the surface representation, we set up an equivalence relation by defining the "form factor" of the curves

$$T^i(x, \Sigma) = \int_{\gamma} dy^i \delta^{(2)}(\vec{x} - \vec{y}), \quad (39)$$

and state that two curves  $\gamma, \gamma'$  are equivalent if they share the same form factor. Every equivalence class defines what we shall call a path. The composition of curves, together with the equivalence relation defines a group product among classes of equivalence, i.e., among paths. It can be shown that this group is Abelian [1]. Now, let us consider path-dependent functionals  $\Psi(\gamma)$  [we employ the same notation for curves and paths]. We introduce the path derivative  $\delta_i(x)$ , that measures the change in  $\Psi(\gamma)$  when an “infinitesimal” path  $u_{\vec{x}}$  is attached to the argument  $\gamma$  of  $\Psi(\gamma)$  at the point  $x$ , up to first order in the vector  $\vec{u}$  associated to the path we have

$$\Psi(\gamma \cdot u_{\vec{x}}) = \Psi(\gamma) + u^i \delta_i(\vec{x}) \Psi(\gamma). \quad (40)$$

One also has a loop derivative [1]  $\Delta_{ij}(\vec{x})$  defined as

$$\Psi(\sigma \cdot C) = \left(1 + \sigma^{ij} \Delta_{ij}(\vec{x})\right) \Psi(C), \quad (41)$$

with  $C$  being a closed path (a loop) and  $\sigma^{ij}$  being the area enclosed by an infinitesimal loop attached at the spatial point  $\vec{x}$ . Thus  $\Delta_{ij}(\vec{x})$  measures how the loop dependent function  $\Psi(C)$  changes under a small deformation of its argument  $C$ . The loop derivative is readily seen to be the curl of the path derivative [15],

$$\Delta_{ij}(\vec{x}) = \frac{\partial}{\partial x^i} \delta_j(\vec{x}) - \frac{\partial}{\partial x^j} \delta_i(\vec{x}). \quad (42)$$

The canonical algebra (37) can be realized by means of the prescriptions

$$\begin{aligned} \hat{A}_i(\vec{x}) &\longrightarrow i \delta_i(\vec{x}), \\ \hat{\Pi}^i(\vec{x}) &\longrightarrow T^i(\vec{x}, \gamma), \\ \hat{r}^i &\longrightarrow r^i, \quad \hat{p}_j \longrightarrow -i \frac{\partial}{\partial r^j} \\ \hat{R}^i &\longrightarrow R^i, \quad \hat{P}_j \longrightarrow -i \frac{\partial}{\partial R^j} \end{aligned} \quad (43)$$

These operators act onto wave functionals  $\Psi[\gamma, \vec{r}, \vec{R}]$  that depend on the path  $\gamma$  and the positions of both types of particles  $\vec{r}, \vec{R}$ . To show that the commutation relations are satisfied it is necessary to use

$$\delta_i(\vec{x}) T^j(\vec{y}, \gamma) = \delta_i^j \delta^{(2)}(\vec{x} - \vec{y}), \quad (44)$$

which can be readily verified.

Using (43) we can write down the covariant momenta as

$$\hat{p}_i - q\hat{A}_i(\vec{r}) \longrightarrow -i\left(\frac{\partial}{\partial r^i} + q\delta_i(\vec{r})\right) \equiv -iD_i(\vec{r}), \quad (45)$$

$$\hat{P}_i - Q\hat{B}_i(\vec{R}) \longrightarrow -i\left(\frac{\partial}{\partial R^i} + iQ\epsilon_{ij}T^j(\vec{R}, \gamma)\right). \quad (46)$$

The gauge invariant combination  $D_i(\vec{r})$  coincides with the path derivative introduced by Mandelstam many years ago [16]. It comprises the ordinary derivative, representing the momentum operator of the particle, plus  $q$  times the “path derivative”  $\delta_i(\vec{r})$ . With these realizations, the physical constraints (38) are written as

$$\begin{aligned} -\left(\rho_1(x) + \partial_i \Pi^i(x, \gamma)\right) \Psi[\gamma, \vec{r}, \vec{R}] &\approx 0 \implies \\ \left(\delta^{(2)}(\vec{x} - \vec{r}) - \sum_s (\delta^{(2)}(\vec{x} - \vec{b}_s) - \delta^{(2)}(\vec{x} - \vec{a}_s))\right) \Psi[\gamma, \vec{r}, \vec{R}] &= 0, \end{aligned} \quad (47)$$

and

$$\begin{aligned} -\left(\rho_2(x) + \epsilon^{ij} \partial_i A_j(x)\right) \Psi[\gamma, \vec{r}, \vec{R}] &\approx 0 \implies \\ \left(\rho_2(x) + \frac{i}{2} \epsilon^{ij} \Delta_{ij}(\vec{x})\right) \Psi[\gamma, \vec{r}, \vec{R}] &= 0. \end{aligned} \quad (48)$$

To write (47) we have used

$$\partial_i T^i(x, \gamma) \equiv -\varrho(\vec{x}, \gamma) = -\sum_s (\delta^{(2)}(\vec{x} - \vec{b}_s) - \delta^{(2)}(\vec{x} - \vec{a}_s)), \quad (49)$$

with  $\vec{a}_s$  and  $\vec{b}_s$  labelling the starting and ending points of the  $s$ -th “strand” of the path, respectively.

To solve the constraint (47) we can use the geometrical setting that allows to solve the Gauss constraint in Maxwell theory coupled to non-relativistic particles [3]. Following this case, we consider “Faraday’s lines” states, consisting on functionals that depend on an open path composed of  $n$  strands starting (or ending) at the particle’s position  $\vec{r}$ . These strands end (or start) at spatial infinity. To take into account the source-free sector, this open strands might be accompanied by closed contours too. For example dropping the contribution arising from the starting points of the strands, the Gauss law constraint (47) can be written as,

$$\begin{aligned} \left(q\delta^{(2)}(\vec{x} - \vec{r}) - \sum_s \delta^{(2)}(\vec{x} - \vec{a}_s)\right) \Psi[\gamma, \vec{r}, \vec{R}] &= 0 \rightarrow \\ \left(q\delta^{(2)}(\vec{x} - \vec{r}) - n\delta^{(2)}(\vec{x} - \vec{r})\right) \Psi[\gamma, \vec{r}, \vec{R}] &= 0. \end{aligned} \quad (50)$$

This equation becomes an identity if  $q = n$  for these incoming paths (analogously  $q = -n$  for outgoing ones). Is it easy to see that for  $N$  charges, one must take  $N$  “bundles” of open paths, one for each charged particle, having as many oriented strands so the sum of incoming minus outgoing strands give the value of the charge. Within this formalism there is no room for fractionary charges, because a Faraday’s line carries a unit of electric flux, which must be emitted from or absorbed by an integral charge  $q$ . We find it convenient to denote the path-dependent functionals that satisfy the Gauss constraint as  $\Psi[\gamma_{\vec{r}}, \vec{R}]$ , since this notation displays both the path and point-dependence and recalls that from now on particles of ”type one” are attached to paths.

It should be observed that Gauge invariant operators respect the geometrical properties of the Faradays lines construction. For instance, the ”covariant momentum”  $-iD_i(\vec{r})$  measures the change of the wave-functional when both the particle and its attached “bundle” of paths are infinitesimally displaced [3].

Once the Gauss constraint (47) is solved, we focus ourselves in the second one (48). Since  $\mathbf{B}_{\vec{A}} = -\frac{1}{2}\epsilon^{ij}F_{ij} \rightarrow -\frac{i}{2}\epsilon^{ij}\Delta_{ij}(\vec{x})$ , this constraint tells us that each particle of ”type two”, whose position is  $\vec{R}$ , carries an amount of “magnetic flux” proportional to its electric charge, and confined to the point where the particle “lives” [3]. We recognize this first class constraint as the one that appears in the Maxwell-Chern-Simons theory when it is quantized in the path representation [11]. Also, this constraint appears in the path formulation of the theory of particles interacting through a Chern-Simons field [3].

Following references [11] and [3], we can then write the solution of (48) as

$$\Psi[\gamma_{\vec{r}}, \vec{R}] = \exp\left(-i\frac{Q}{2\pi}\Theta(\gamma_{\vec{r}}, \vec{R})\right) \Phi(\partial\gamma_{\vec{r}}, \vec{R}), \quad (51)$$

with  $\epsilon^{ij}\Delta_{ij}(\vec{x})\Phi(\partial\gamma_{\vec{r}}, \vec{R}) = 0$  and

$$\frac{1}{2}\epsilon^{ij}\Delta_{ij}(\vec{x})\Theta(\gamma_{\vec{r}}, \vec{R}) = \rho_2(x) \quad (52)$$

The condition on  $\Phi(\partial\gamma_{\vec{r}}, \vec{R})$  forces it to be a function that depends on the path  $\gamma_{\vec{r}}$  only through its boundary  $\partial\gamma_{\vec{r}} = \vec{r}$ . The solution for  $\Theta(\gamma_{\vec{r}}, \vec{R})$  is the algebraic sum of the angles subtended by the pieces (the strands) of the path  $\gamma$ , measured from the point,  $\vec{R}$ , where the ”big” particle is

$$\Theta(\gamma_{\vec{r}}, \vec{R}) \equiv \int_{\gamma} dx^j \varepsilon^{ij} \left[ \frac{(x - R)^i}{|\vec{x} - \vec{R}|^2} \right]. \quad (53)$$

It is interesting to remark a mayor difference between this case and both the particles-Chern-Simons [3] and Maxwell-Chern-Simons [11] cases, concerning the constraint (48). The present case does not suffers from what could be called the "self-angle" problem. By this we refer to the fact that in expression (53), the angle subtended by the paths is measured with respect to points that do not coincide with the ending points of the paths. This contrasts with the Maxwell-CS and particles-CS cases, where there appear self-interaction effects that in the path representation manifest through the dependence of the wave functional on the angle subtended by the path measured with respect to its own endpoints. This "self-angle" is ill defined, and requires some regularizing prescription to deal with it [18].

At this point one should verify whether the gauge invariant operators of the theory preserve the form of the physical states  $\Psi[\gamma_{\vec{r}}, \vec{R}] = \exp\left(-i\frac{Q}{2\pi}\Theta(\gamma_{\vec{r}}, \vec{R})\right) \Phi(\vec{r}, \vec{R})$ . For instance, let us consider the action of the Mandelstam derivative onto the gauge invariant states. It is given by

$$\begin{aligned} -iD_i(\vec{r})\Psi_{Phys} &= -iD_i(\vec{r}) \left[ \exp\left(-i\frac{Q}{2\pi}\Theta(\gamma_{\vec{r}}, \vec{R})\right) \Phi(\vec{r}, \vec{R}) \right] \\ &= \exp\left(-i\frac{Q}{2\pi}\Theta(\gamma_{\vec{r}}, \vec{R})\right) \left[ -i\frac{\partial}{\partial r^i} + \frac{qQ}{2\pi}\epsilon_{ij}\frac{(r-R)^j}{|\vec{r}-\vec{R}|^2} \right] \Phi(\vec{r}, \vec{R}) \\ &= \exp\left(-i\frac{Q}{2\pi}\Theta(\gamma_{\vec{r}}, \vec{R})\right) \Phi'(\vec{r}, \vec{R}), \end{aligned} \quad (54)$$

where the second line defines  $\Phi'(\vec{r}, \vec{R})$ . Hence, we see that the Mandelstam derivative leaves invariant the physical space of states, as it should be.

On the other hand, it can be verified that the other "covariant momentum"

$$P_i - QB_i(\vec{R}) \rightarrow -i\left(\frac{\partial}{\partial R^i} + iQ\epsilon_{ij}T^j(\vec{R}, \gamma)\right), \quad (55)$$

produces a result analogous to (54) when applied to the physical functionals

$$\begin{aligned} -i\left(\frac{\partial}{\partial R^i} + iQ\epsilon_{ij}T^j(\vec{R}, \gamma)\right)\Psi_{Phys} &= \\ &= \exp\left(-i\frac{Q}{2\pi}\Theta(\gamma_{\vec{r}}, \vec{R})\right) \left[ -i\frac{\partial}{\partial R^i} - \frac{qQ}{2\pi}\epsilon_{ij}\frac{(r-R)^j}{|\vec{r}-\vec{R}|^2} \right] \Phi(\vec{r}, \vec{R}) \\ &= \exp\left(-i\frac{Q}{2\pi}\Theta(\gamma_{\vec{r}}, \vec{R})\right) \Phi''(\vec{r}, \vec{R}), \end{aligned} \quad (56)$$

where we have taken into account the quantization condition (50) for q. Again, gauge invariance is maintained.

As in the Maxwell-Chern-Simons and particles-Chern-Simons cases [3, 11] there is a unitary transformation that allows us to eliminate the path dependent phase factor  $\exp\left(-i\frac{Q}{2\pi}\Theta(\gamma_{\vec{r}}, \vec{R})\right)$ . Once this transformation is performed, the path dependence of the wave functional is reduced to the boundary of the path  $\partial\gamma_{\vec{r}}$ , which is just the set of the positions  $\{\vec{r}\}$  of the type-one particles. At this point, the boundary dependence of the wave functional becomes redundant, and it suffices to employ ordinary (i.e., point-dependent) wave functions  $\Psi(\vec{r}, \vec{R})$ . At the same time it is not necessary to maintain the path (or loop) derivatives in the physical operators, and we may substitute them by ordinary derivatives[3].

Once this unitary transformation is performed, the Schrödinger equation of the model can be written down as

$$i\partial_t\Psi(\vec{r}, t) = H_0\Psi(\vec{r}, t), \quad (57)$$

where the Hamiltonian  $H_0$  is

$$H_0 = \frac{mv^2}{2} + \frac{MV^2}{2}, \quad (58)$$

where  $mv^i$  and  $MV^i$  act on the states as

$$mv^i = -i\frac{\partial}{\partial r^i} - eqA_i(\vec{r}) = p_i + \frac{qQ}{2\pi}\epsilon_{ij}\frac{(r^j - R^j)}{|\vec{r} - \vec{R}|^2} \quad (59)$$

$$MV^i = -i\frac{\partial}{\partial R^i} - QB_i(\vec{R}) = P_i - \frac{qQ}{2\pi}\epsilon_{ij}\frac{(r^j - R^j)}{|\vec{r} - \vec{R}|^2}. \quad (60)$$

Thus we arrive to the quantum mechanics of two species of non-relativistic particles that interact through potentials that satisfy

$$qA_i(\vec{r}) = -QB_i(\vec{R}) = -\frac{qQ}{2\pi}\epsilon_{ij}\frac{(r^j - R^j)}{|\vec{r} - \vec{R}|^2} \quad (61)$$

It is straightforward to see that this potentials solve the constraints  $\chi_1(x)$  and  $\chi_2(x)$  as in (35). This long-range interaction coincides with the topological interaction experienced by anyons [7, 8, 9, 10, 19]. In fact, we have recovered the hamiltonian of precisely two anyons, which is exactly soluble [6, 7, 8, 9].

It is worth recalling that the equations describing anyons can be rewritten in what some authors call the "anyon gauge". It is obtained by performing a singular gauge transformation that converts the Schrodinger equation for the topologically interacting particles into that of a free-particles system. However, the interaction remains hidden in the fact that the wave function becomes multivalued. A moment of reflection allows to see that this features are



neatly realized in the path-dependent formulation: the wave function in the anyon gauge

$$\Psi(\vec{r}, t) = \exp\left(-i\frac{Q}{2\pi}\Theta(\gamma_{\vec{r}}, \vec{R})\right)\Phi(\vec{r}, t), \quad (62)$$

precisely corresponds to the multivalued wave function (51), while the Mandelstam derivative is just the "partial derivative" appropriate to act onto that multivalued wave function, that turned to be a path-dependent one.

#### IV. 3 + 1 DIMENSIONAL BF THEORY AND NON-RELATIVISTIC STRING-PARTICLE INTERACTION

Our last model consists on a dynamical string interacting with a dynamical particle by means of a topological BF term in 3 + 1 dimensions. Both string and particle are non-relativistic, and are described as in the preceding sections. For the sake of simplicity we restrict ourselves to consider just one particle and one string, although the formulation could certainly be extended to a more general case. The BF term is analogous to its counterpart in 2 + 1 dimensions studied in the last section. We take the action as

$$\begin{aligned} S = & \frac{1}{4} \int d^4x \epsilon^{\mu\nu\lambda\rho} B_{\mu\nu} F_{\lambda\rho} + \int dt \left( \frac{1}{2} m \dot{\vec{r}}^2 \right) + \frac{\alpha}{2} \int dt \int d\sigma \left[ (\dot{z}^i)^2 - (z'^i)^2 \right] \\ & + \int d^4x \left( J^\mu A_\mu + \frac{1}{2} J^{\mu\nu} B_{\mu\nu} \right). \end{aligned} \quad (63)$$

As before, we have

$$J^\mu(\vec{x}, t) = q \int dy^\mu \delta^{(4)}(x - y) = q v^\mu(t) \delta^{(3)}(\vec{x} - \vec{r}) \equiv (\rho(x), \vec{J}(x)), \quad (64)$$

$$J^{\mu\nu}(\vec{x}, t) = \phi \int dt \int d\sigma [\dot{z}^\mu z'^\nu - \dot{z}^\nu z'^\mu] \delta^{(4)}(x - z), \quad (65)$$

where  $v^\mu(t) = (1, \vec{v})$ , and  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ . As in the preceeding sections,  $\alpha$  is the string tension, and  $\phi$  and  $q$  are dimensionless, so the units of the fields are clear in this context.

The 3 + 1 decomposition of the action yields

$$\begin{aligned} S = & \frac{1}{2} \int d^4x \left[ \dot{A}_i \epsilon^{ijk} B_{jk} + B_{0i} \epsilon^{ijk} F_{jk} + A_0 \epsilon^{ijk} \partial_i B_{jk} \right] \\ & + \int dt \left( \frac{1}{2} m \dot{\vec{r}}^2 \right) + \frac{\alpha}{2} \int dt \int d\sigma \left[ (\dot{z}^i)^2 - (z'^i)^2 \right] \\ & + q \int dt \left[ \dot{r}^i(t) A_i(\vec{r}(t)) + A_0(\vec{r}(t)) \right] + \phi \int dt \int d\sigma \left[ B_{ij}(z(\sigma, t)) \dot{z}^i z'^j + B_{0k}(z(\sigma, t)) \dot{z}^k \right], \end{aligned}$$

$[z^0 = t, \dot{z}^0 = 1, z'^0 = 0]$ . The expressions  $A_\mu(\vec{r}(t), t) \equiv A_\mu(\vec{r})$  and  $B_{\mu\nu}(z(\sigma, t)) \equiv B_{\mu\nu}(z)$  should be understood as a shorthand for

$$A_\mu(\vec{r}, t) \equiv \int d^3\vec{x} \delta^3(\vec{x} - \vec{r}) A_\mu(\vec{x}, t), \quad (66)$$

$$B_{\mu\nu}(z(\sigma, t)) \equiv \int d^3\vec{x} \delta^3(\vec{x} - \vec{z}) B_{\mu\nu}(\vec{x}, t). \quad (67)$$

We now summarize the Dirac quantization of the theory. The conjugate momenta associated to the particle and string are given by

$$\Pi^k_{(A)} = \frac{1}{2} \epsilon^{ijk} B_{ij}, \quad (68)$$

$$p_i = m \frac{dr^i}{dt} + q A_i(\vec{r}) \longrightarrow \dot{r}^i = \frac{(p_i - q A_i(\vec{r}))}{m}, \quad (69)$$

$$P_{i(z)} = \alpha \dot{z}^i + \phi B_{ij}(z) z'^j \longrightarrow \dot{z}^i = \frac{(P_{i(z)} - \phi B_{ij}(z) z'^j)}{\alpha}. \quad (70)$$

On the other hand, and as in the preceding section, it should be noticed that  $A_i$  and  $\frac{1}{2} \epsilon^{ijk} B_{jk}$  are already canonical conjugate variables. Also, we will not take  $B_{i0}$  and  $A_0$  as dynamical variables from the beginning.

The Hamiltonian can be written as follows

$$\begin{aligned} H &= \frac{(p_i - q A_i(\vec{r}))^2}{2m} + \int d\sigma \frac{\alpha}{2} \left[ \frac{1}{\alpha^2} (P_{i(z)} - \phi B_{ij}(z) z'^j)^2 + (z'^i)^2 \right] + \int d^3x A_0(x) \chi(x) \\ &\quad + \int d^3x B_{0j}(x) \chi^j(x) \\ &\equiv H_0 + \int d^3x A_0(x) \chi(x) + \int d^3x B_{0j}(x) \chi^j(x). \end{aligned} \quad (71)$$

So  $B_{i0}$  and  $A_0$  appear as the Lagrange multipliers associated to the first class constraints that generate time independent gauge transformations

$$\chi(x) \equiv -\rho(x) - \frac{1}{2} \epsilon^{ijk} \partial_i B_{jk} \quad (72)$$

$$\chi^i(x) \equiv -\rho^i(x) - \epsilon^{ijk} \partial_j A_k(x), \quad (73)$$

with

$$\rho(x) = q \delta^{(3)}(\vec{x} - \vec{r}), \quad (74)$$

$$\rho^i(x) \equiv \phi \int d\sigma z'^i \delta^{(3)}(\vec{x} - \vec{z}). \quad (75)$$

The canonical Poisson brackets are defined as

$$\{r^i, P_j\} = \delta_j^i, \quad (76)$$

$$\{z^i(\sigma), P_j(\sigma')\} = \delta_j^i \delta(\sigma - \sigma'), \quad (77)$$

$$\left\{A_i(\vec{x}), \frac{1}{2}\epsilon^{jkl}B_{kl}(\vec{y})\right\} = \delta_i^j \delta^{(3)}(\vec{x} - \vec{y}). \quad (78)$$

The remaining Poisson brackets vanish. Due to the topological character of the BF term the contribution of the fields to the Hamiltonian looks as if they were external fields, just like in the preceding section. The basic observables, in Dirac's sense, are the positions of the particle and string,  $\vec{r}$  and  $\vec{z}(\sigma)$ , and the "covariant" (gauge invariant) momenta of the particle and the string, given by

$$p_i - qA_i(\vec{r}), \quad P_{i(z)} - \phi B_{ij}(z)z'^j, \quad (79)$$

respectively. All the physical observables of the theory are built in terms of these gauge invariant quantities. For instance, the Hamiltonian fulfils this rule.

To quantize, the canonical variables are promoted to operators obeying the commutators that result from the usual replacement  $\{, \} \longrightarrow -i[, ]$ . These operators have to be realized in a Hilbert space of physical states  $|\Psi\rangle_{Phys}$  that obey

$$-\left(\rho(x) + \frac{1}{2}\epsilon^{ijk}\partial_i B_{jk}\right)|\Psi\rangle_{Phys} \approx 0 \quad (80)$$

$$-\left(\rho^i(x) + \epsilon^{ijk}\partial_j A_k(x)\right)|\Psi\rangle_{Phys} \approx 0 \quad (81)$$

Now we seek for a geometric representation appropriate to the present model. As we shall discuss, there are two possible choices, depending on which of the field operators ( $A_\mu(x)$  or  $B_{\mu\nu}(x)$ ) we take as "position" or as "derivative" operator. In both choices, we shall realize the operators associated to the particle and string in a Schrodinger or "shape" representation, i.e., we shall take

$$\hat{r}^i \longrightarrow r^i, \quad \hat{z}^i(\sigma) \longrightarrow z^i(\sigma), \quad (82)$$

$$\hat{P}^i \longrightarrow -i\frac{\partial}{\partial r^i}, \quad \hat{P}^i(\sigma) \longrightarrow -i\frac{\delta}{\delta z^i(\sigma)}. \quad (83)$$

These operators are supposed to act onto functionals  $\Psi[\vec{r}, z(\sigma)]$  that depend on the coordinates of the particle and of the string world-sheet. Once the particle and string operators

are realized, we have also to accommodate the fields operators into the description. The first geometric representation that we are going to consider is a “Faraday’s lines” or path representation [1]. We begin observing that the fundamental commutator associated to equation (78) can be realized on path-dependent functionals if one prescribes

$$\begin{aligned}\hat{A}_i(\vec{x}) &\longrightarrow i\delta_i(\vec{x}), \\ \hat{\Pi}^i &= \frac{1}{2}\epsilon^{ijk}\hat{B}_{jk}(\vec{x}) \longrightarrow T^i(\vec{x}, \gamma),\end{aligned}\tag{84}$$

where  $\delta_i(\vec{x})$  and  $T^i(\vec{x}, \gamma)$  were defined in (39) and (40). In this representation we can write

$$\begin{aligned}\hat{p}_i - q\hat{A}_i(\vec{r}) &\longrightarrow -i\left(\frac{\partial}{\partial r^i} + q\delta_i(\vec{r})\right) \equiv -iD_i(\vec{r}), \\ \hat{P}_{i(z)} - \phi\hat{B}_{ij}(z)\hat{z}'^j &\longrightarrow -i\left(\frac{\delta}{\delta z^i(\sigma)} - i\phi\epsilon_{ijk}z'^jT^k(z(\sigma), \gamma)\right),\end{aligned}\tag{85}$$

where  $D_i(\vec{x})$  is the “Mandelstam operator” as defined in (45). Thus, the states of the interacting theory can be taken as functionals  $\Psi[\vec{r}, z(\sigma), \gamma]$ . Among them, the physical ones will be those that satisfy the constraints (80) and (81). The former can be written down as

$$\begin{aligned}-\left(\rho(x) + \frac{1}{2}\epsilon^{ijk}\partial_i B_{jk}\right)\Psi[\vec{r}, z(\sigma), \gamma] &\approx 0 \quad \implies \\ \left(\rho(x) + \partial_i T^i(\vec{x}, \gamma)\right)\Psi[\vec{r}, z(\sigma), \gamma] &= \left(q\delta^{(3)}(\vec{x} - \vec{r}) - \varrho(\vec{x}, \gamma)\right)\Psi[\vec{r}, z(\sigma), \gamma] = 0,\end{aligned}\tag{86}$$

where  $\varrho(\vec{x}, \gamma)$  was defined before in (49).

Regarding the second constraint, we have

$$\begin{aligned}-\left(\rho^i(x) + \epsilon^{ijk}\partial_j A_k(x)\right)\Psi[\vec{r}, z(\sigma), \gamma] &\approx 0 \implies \\ \left(\rho^i(x) + i\frac{1}{2}\epsilon^{ijk}\Delta_{jk}(\vec{x})\right)\Psi[\vec{r}, z(\sigma), \gamma] &= 0,\end{aligned}\tag{87}$$

with  $\Delta_{ij}(\vec{x})$  given in section III. At this point it will be useful to recall the solution of the constraints in the “toy model”. We see that (86) is similar to (47), while (87) corresponds to a generalized version of (48). From our experience with those constraints, we obtain the following picture: (86) tells us that we have to take as physical wave functions those that depend on an open-path of  $n$ -strands that meet at the point  $\vec{r}$  where the charged particle is located (as before, this open path may also comprise closed pieces). The number of oriented strands “sum up” to the (quantized) value of the electric charge. These “Faradays lines” drawing has to be accompanied by a closed string, which has nothing attached in this

representation. From now on we shall write  $\gamma_{\vec{r}}$  instead of  $\gamma$ , because gauge invariance joins paths and particles as explained before. On the other hand, following the solution of (48) we see that (87) obligates to write the physical wave functionals as

$$\Psi[\vec{r}, \vec{z}(\sigma), \gamma_{\vec{r}}] = \exp(i \Theta(\vec{r}, z(\sigma), \gamma_{\vec{r}})) \Phi(\vec{r}, \vec{z}(\sigma)), \quad (88)$$

with

$$\frac{1}{2} \epsilon^{ijk} \Delta_{jk}(\vec{x}) \Theta(\vec{r}, z(\sigma), \gamma_{\vec{r}}) = \rho^i(\vec{x}). \quad (89)$$

The solution of (89) is

$$\Theta(\vec{r}, z(\sigma), \gamma_{\vec{r}}) = \frac{\phi}{4\pi} \int_{\gamma} dx^i \int_{\Gamma} d\sigma z'^j \epsilon_{ijk} \frac{(x - z(\sigma))^k}{|\vec{x} - \vec{z}(\sigma)|^3}, \quad (90)$$

where  $\gamma$  is the path (as usual), and  $\Gamma$  is the closed curve that coincides with the closed string.

Thus, we obtain that the physical wave function comprises a fixed path-dependent phase factor times  $\Phi$ , which depends on the path  $\gamma$  only through its ending point (the one which is not at spatial infinity), that is where the particle “lives”. In order to give a physical interpretation of  $\Theta(\gamma)$ , we use Stokes Theorem and the fact that

$$\frac{(x - z(\sigma))^k}{|\vec{x} - \vec{z}(\sigma)|} = \frac{\partial}{\partial z^k} \left[ \frac{1}{|\vec{x} - \vec{z}(\sigma)|} \right] = - \frac{\partial}{\partial x^k} \left[ \frac{1}{|\vec{x} - \vec{z}(\sigma)|} \right] \quad (91)$$

to rewrite (90) in the form

$$\Theta(\vec{r}, z(\sigma), \gamma_{\vec{r}}) = \phi \left[ \frac{1}{4\pi} \int_{\Sigma(\Gamma)} dS_i \left[ \frac{(b - z)^i}{|\vec{b} - \vec{z}|^3} - \frac{(a - z)^i}{|\vec{a} - \vec{z}|^3} \right] - \int_{\Sigma(\Gamma)} dS_i \int_{\gamma} dx^i \delta^{(3)}(\vec{x} - \vec{z}) \right], \quad (92)$$

with  $dS_i \equiv \epsilon_{ijk} d\Sigma^{jk}$  ( $d\Sigma^{jk}$  was defined in eq. (16)). In the last expression  $\Sigma(\Gamma)$  is an open surface that has the string  $\Gamma$  as its border, i.e.,  $\partial\Sigma(\Gamma) = \Gamma$ . The first term in (92) is the solid angle subtended by the surface  $\Sigma(\Gamma)$  attached to the string measured from the final point  $\vec{b}$ , minus the solid angle subtended by the same surface but measured from the starting point  $\vec{a}$  of the path  $\gamma$  [again, it should be recalled that one of these points is at spatial infinity]. In turn, the second term in (92) counts the number of times that the path  $\gamma$  intersects the surface  $\Sigma(\Gamma)$ . Although  $\Theta(\vec{r}, z(\sigma), \gamma_{\vec{r}})$ , given in equation (92) looks like a surface-dependent quantity, this dependence is only apparent, as can be realized by just turning back to (90).

It is worth comparing this case with the toy model of the preceding section. In the toy model the geometrical phase analogous to  $\Theta(\vec{r}, z(\sigma), \gamma_{\vec{r}})$  measured the winding number of

the path attached to one of the particles, around the other particle in the plane. In the present case,  $\Theta(\vec{r}, z(\sigma), \gamma_{\vec{r}})$  generalizes this geometrical fact to a three dimensional situation: it measures the "winding" of the path attached to the particle "around" the closed string. There is yet another possibility of generalizing this in three dimensions. It corresponds precisely to the other geometric representation, that we next discuss.

The second, "dual" representation is, in fact, a surface-dependent representation, as the one discussed in section II. We set

$$\begin{aligned}\hat{A}_i(\vec{x}) &\longrightarrow -\frac{1}{2}\epsilon_{ijk}T^{jk}(\vec{x}, \Sigma), \\ \hat{\Pi}^i(\vec{x}) &\longrightarrow i\epsilon^{ijk}\delta_{jk}(\vec{x}) \implies \\ \hat{B}_{ij}(\vec{x}) &\longrightarrow 2i\delta_{ij}(\vec{x}),\end{aligned}\tag{93}$$

where  $T^{jk}(\vec{x}, \Sigma)$  and  $\delta_{ij}(\vec{x})$  were defined in (16) and (17). It can be readily seen that this prescriptions realize the canonical commutators of the theory. Also, a straightforward calculation from (72), using (93) and (19) leads to write the first class constraints as,

$$\chi(\vec{x}) = -\rho(\vec{x}) - \frac{i}{3}\epsilon^{ijk}\Delta_{ijk} = 0,\tag{94}$$

and

$$\begin{aligned}\chi^i(\vec{x}) &= -\rho^i(\vec{x}) + \epsilon^{ijk}\partial_j\left(\frac{1}{2}\epsilon_{klm}T^{lm}(\vec{x}, \Sigma)\right) = -\rho^i(\vec{x}) + T^i(\vec{x}, \partial\Sigma) = 0 \longrightarrow \\ &-\phi\int_{\Gamma} dz^k\delta^{(3)}(\vec{x} - \vec{z}_{(\sigma)}) + \int_{\gamma=\partial\Sigma} dy^k\delta^{(3)}(\vec{x} - \vec{y}) = 0.\end{aligned}\tag{95}$$

Equation (95) tells us that the string should coincide with the boundary of the open surface (see [3] and the discussions of section II). On the other hand, our previous experience with the toy model and with the former representation for the present model also teaches us that the solution of (94) is given by wave functionals of the form (see [3, 11] and the discussion of section III)

$$\Psi[\vec{r}, z, \Sigma] = \exp\left(i\Theta(\vec{r}, \vec{z}, \Sigma)\right)\Phi(\vec{r}, \vec{z}),\tag{96}$$

where

$$\Theta(\vec{r}, \vec{z}, \Sigma) = \frac{q}{8\pi}\int_{\Sigma(\Gamma)} dS_{iy}\frac{(y-r)^i}{|\vec{y}-\vec{r}|^3},\tag{97}$$

is proportional to the solid angle subtended by the surface  $\Sigma(\Gamma)$  measured from the position of the particle  $\vec{r}$ . In analogy with the toy model and with the path representation of this

model, we see that the dependence on the surface is restricted to a phase factor, which measures a topological feature: how many times the surface attached to the string wraps around the particle.

So, in the surface representation we end up with strings having a bundle of  $n$  pieces of open-surfaces attached to them, with  $n$  depending of the value of the quantized constant  $\phi$  ( i.e., the “charge” of the string). Also, the wave functional depends of a “lonely” point charged particle. The role of the surface is to take into account how the the string and particle are topologically related. It could be said that the difference between the two dual representations is encoded in the following feature: which of the matter objects (the particle or the string) is left alone, and which has an attached object whose winding or wrapping around the other carries the content of the topological interaction.

It is interesting to see how gauge invariance is maintained through a geometric mechanism, in both representations . For instance, in the path representation (84), the “covariant” momentum associated with the particle (79) is again realized as a “Mandelstam” operator that translates both the particle and its attached ”bundle of paths” together (see discussion in section III). Also, in the surface representation (93) where the string is coupled to the ”bundle of surfaces”, the covariant momentum of the string (expression (79)) translates both the string and the set of surfaces together, thus maintaining the geometrical picture dictated by gauge invariance (see also the discussion at the end of the first section)

$$P_{i(z)} - \phi B_{ij}(z) z'^j \longrightarrow -i \left( \frac{\delta}{\delta z^i} + 2\phi \delta_{ij}(\vec{z}) z'^j \right). \quad (98)$$

The last expression is a kind of generalized “Mandelstam operator” for the string-surface representation.

On the other hand, gauge invariance also restricts the form of the path (or surface) dependent wave functional, accordingly with (88) or (96). We should check that the observables of the theory respect this particular form. To this end, we apply the gauge invariant momenta to the physical states  $\Psi_{Pys}$ . In the path representation we obtain

$$\begin{aligned} p_i - q A_i(\vec{r}) &\rightarrow -i \left( \frac{\partial}{\partial r^i} + q \delta_i(\vec{r}) \right) \Psi_{Pys}(\vec{r}, \vec{z}, \gamma_{\vec{r}}) = \\ &= \exp(i \Theta) \left[ -i \frac{\partial}{\partial r^i} + \frac{q\phi}{4\pi} \int_{\Gamma} dz^j \epsilon_{ijk} \frac{(r-z)^k}{|\vec{r}-\vec{z}|^3} \right] \Phi(\vec{r}, \vec{z}) \\ &= \exp [i \Theta(\vec{r}, z(\sigma), \gamma_{\vec{r}})] \times \Phi'(\vec{r}, \vec{z}), \end{aligned} \quad (99)$$

and using the constraint (86) we get

$$\begin{aligned}
P_{i(z)} - \phi B_{ij}(z) z'^j &\rightarrow -i \left( \frac{\delta}{\delta z^i} - i \phi \epsilon_{ijk} z'^j T^k(\vec{z}, \gamma) \right) \Psi_{P_{ys}}(\vec{r}, \vec{z}, \gamma_{\vec{r}}) = \\
&= \exp(i \Theta) \left[ -i \frac{\delta}{\delta z^i} + \frac{q \phi}{4\pi} \epsilon_{ijk} z'^j \frac{(r-z)^k}{|\vec{r}-\vec{z}|^3} \right] \Phi(\vec{r}, \vec{z}) \\
&= \exp [i \Theta(\vec{r}, z(\sigma), \gamma_{\vec{r}})] \times \Phi'(\vec{r}, \vec{z}).
\end{aligned} \tag{100}$$

In turn, in the surface representation we have

$$\begin{aligned}
p_i - q A_i(\vec{r}) &\rightarrow -i \left( \frac{\partial}{\partial r^i} + i \frac{q}{2} \epsilon_{ijk} T^{jk}(\vec{r}, \Sigma) \right) \Psi_{P_{ys}}(\vec{r}, \vec{z}, \Sigma) = \\
&= \exp(i \Theta) \left[ -i \frac{\partial}{\partial r^i} + \frac{q \phi}{4\pi} \int_{\Gamma} dz^j \epsilon_{ijk} \frac{(r-z)^k}{|\vec{r}-\vec{z}|^3} \right] \Phi(\vec{r}, \vec{z}) \\
&= \exp [i \Theta(\vec{r}, z(\sigma), \Sigma)] \times \Phi'(\vec{r}, \vec{z}),
\end{aligned} \tag{101}$$

where we have used the quantization constraint (95). For the other gauge invariant operator we have

$$\begin{aligned}
P_{i(z)} - \phi B_{ij}(z) z'^j &\rightarrow -i \left( \frac{\delta}{\delta z^i} + 2 \phi \delta_{ij}(\vec{z}) z'^j \right) \Psi_{P_{ys}}(\vec{r}, \vec{z}, \Sigma) = \\
&= \exp(i \Theta) \left[ -i \frac{\delta}{\delta z^i} + \frac{q \phi}{4\pi} \epsilon_{ijk} z'^j \frac{(r-z)^k}{|\vec{r}-\vec{z}|^3} \right] \Phi(\vec{r}, \vec{z}) \\
&= \exp [i \Theta(\vec{r}, z(\sigma), \Sigma)] \times \Phi'(\vec{r}, \vec{z}).
\end{aligned} \tag{102}$$

In all these expressions, the functionals "  $\Phi'$  " only depend on the particle and string positions. Hence, the observables leave invariant the physical sector of the Hilbert space, as required. Furthermore, these expressions indicate that the path or the surface dependence may be eliminated by performing a unitary transformation that extracts from the wave functional the phase factor  $\exp [i \Theta(\vec{r}, z(\sigma), \gamma_{\vec{r}})]$  or  $\exp [i \Theta(\vec{r}, z(\sigma), \Sigma)]$ . This unitary transformation appeared in similar contexts in [3, 11].

It is interesting to see how the hamiltonian looks before performing the unitary transformation mentioned above. In the path representation we have, introducing (85) in (71)

$$H_0 = \frac{\left[ -i \left( \frac{\partial}{\partial r^i} + q \delta_i(\vec{r}) \right) \right]^2}{2m} + \int d\sigma \frac{\alpha}{2} \left[ \frac{\left( -i \frac{\delta}{\delta z^i} - \phi \epsilon_{ijk} z'^j T^k(\vec{z}, \gamma) \right)^2}{\alpha^2} + (z'^i)^2 \right]. \tag{103}$$



On the other hand, by a similar calculation, in the surface representation the hamiltonian would be given by

$$H_0 = \frac{\left[-i\left(\frac{\partial}{\partial r^i} + i\frac{q}{2}\epsilon_{ijk}T^{jk}(\vec{r}, \Sigma)\right)\right]^2}{2m} + \int d\sigma \frac{\alpha}{2} \left[ \frac{\left(-i\frac{\delta}{\delta z^i} - 2i\phi\delta_{ij}(\vec{z})z'^j\right)^2}{\alpha^2} + (z'^i)^2 \right]. \quad (104)$$

Both expressions yield the same Schrodinger equation

$$\begin{aligned} i\frac{\partial}{\partial t}\Psi[\vec{r}, \vec{z}] &= H_0\Psi[\vec{r}, \vec{z}] \\ &= \left\{ \frac{1}{2m} \left[ -i\frac{\partial}{\partial r^i} + \frac{q\phi}{4\pi} \int_{\Gamma} dz^j \epsilon_{ijk} \frac{(r-z)^k}{|\vec{r}-\vec{z}|^3} \right]^2 \right. \\ &\quad \left. + \int d\sigma \frac{\alpha}{2} \left[ \frac{\left(-i\frac{\delta}{\delta z^i} + \frac{q\phi}{4\pi} \epsilon_{ijk} z'^k \frac{(r-z)^j}{|\vec{r}-\vec{z}|^3}\right)^2}{\alpha^2} + (z'^i)^2 \right] \right\} \times \Phi(\vec{r}, \vec{z}), \end{aligned} \quad (105)$$

once the unitary transformation is performed. The last equation is the analogous of the equation for the system of two anyons (59) that the toy model yielded. The right hand side corresponds to the energy of a particle and a string that interact through a topological generalized potential of the form

$$A_i(\vec{r}, \vec{z}) = \frac{q\phi}{4\pi} \int_{\Gamma} d\sigma \epsilon_{ijk} z'^j \frac{(r-z)^k}{|\vec{r}-\vec{z}|^3}. \quad (106)$$

This suggests that there should be an equivalent formulation of the model, that only deals with particle and string variables (and not with topological fields) from the very beginning. In fact, it is easy to see that the lagrangean

$$L = \frac{1}{2}m\dot{\vec{r}}^2 + \int d\sigma \frac{\alpha}{2} \left[ (\dot{z}^i)^2 - (z'^i)^2 \right] + \frac{q\phi}{4\pi} \int_{\Gamma} d\sigma \epsilon_{ijk} z'^j \frac{(r-z)^k}{|\vec{r}-\vec{z}|^3} (\dot{r}^i - \dot{z}^i), \quad (107)$$

fulfils this requisite.

## V. DISCUSSION

We have studied the geometric representation of strings interacting by means of the Kalb-Ramond field. We saw that this representation is a “surface representation” that may be set up only if the coupling constant  $\phi$  of the string (equivalent to the charge if they were

point particles) is quantized as integer values  $n$ . This theory is in a sense very similar to the Maxwell theory interacting with 0-dimensional objects studied in the framework of the LR in [3]. In this case the quantization within the LR was a “Faraday’s lines representation” where the quantization of the charges was stated in terms of the fundamental unit of electric flux carried by each Faraday’s line. In both cases the appropriate Hilbert space is made of wave functionals whose arguments are geometric “Faraday’s extended objects” (that in this work are surfaces) emanating from or ending at the strings (or particles) positions.

We also studied two generalizations of the path-space formulation of the theory of particles interacting through a Chern-Simons field [3]. First, we considered the theory of a set of two types of particles coupled to a BF topological term (in  $2 + 1$ -dimensions). Although this theory has an interest on its own, because it has a direct relationship with the problem of interacting anyons, it also serves to prepare the scene for the study carried out in the last section, where we consider a model that involves extended objects (strings) and particles interacting through a BF term in  $3+1$  dimensions. In both models, quantization of the corresponding “charge” of the material objects involved (i.e., point particles or strings) is necessary for the consistence of the geometric representation.

Also, both models share the following feature: the topological interaction can be casted into a kind of multivaluedness of the corresponding wave functionals, that in the geometrical representation is manifested through the functional dependence on the winding number of a path around a point in the plane (in the  $2 + 1$  dimensions case), the winding number of a path around a closed string, or the wrapping number of a surface around a point (in the  $3 + 1$  BF model).

To conclude, it is interesting to point out that, as in the  $2 + 1$  model of anyons, it is possible to decouple the center of mass and the relative motions in the particle-string model through the introduction of the variables

$$\vec{r}_{rel} = \vec{r} - \vec{z}(\sigma); \quad \vec{R}_{CM} = \frac{m\vec{r} + \alpha \int \vec{z}(\sigma) d\sigma}{m + \alpha}. \quad (108)$$

Then, the lagrangian can be alternatively written down as

$$L = \frac{1}{2}m\dot{\vec{R}}_{CM}^2 + \frac{\alpha}{2} \int (\dot{\vec{r}}_{rel}^2 - \dot{\vec{r}}_{rel}^2) d\sigma - \frac{\alpha^2}{2(m + \alpha)} \left( \int \dot{\vec{r}}_{rel}^2 d\sigma \right)^2 - \frac{q\phi}{4\pi} \dot{\Omega}. \quad (109)$$

Hence, in the particle-string model, the topological interaction contributes to the lagrangian as the total derivative of a multivalued function, namely, the solid angle subtended by the

string measured from the particles position. This feature is a nice generalization of what occurs in its relative  $2 + 1$  dimensional model of two anyons, and we believe that their consequences deserve to be further considered [20].

## VI. ACKNOWLEDGMENTS

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